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$\Psi = We^{\pm\Phi}$ Quantum Cosmological Solutions for Class A Bianchi Models

O. Obregón[1,2] and J. Socorro[1]

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We find exact solutions to the Wheeler–DeWitt equation, for a certain factor ordering. They have the form $\Psi = We^{\pm\Phi}$ for class A Bianchi models, where Φ is a solution to the classical Hamilton–Jacobi equation, generalizing the only known solution of Moncrief and Ryan for the Bianchi type IX model in standard quantum cosmology. The same kind of solution has also been found in supersymmetric quantum cosmology.

1. INTRODUCTION

In recent years, progress has been made in finding solutions (Ashtekar, 1986; Brügmann *et al.*, 1992) to the canonical constraints of the full general relativity theory. However, the canonical quantization program is far from complete. It is hoped that the study of some particular models could illustrate the behavior of the general theory. The Bianchi cosmologies are the prime example. Even in these simplified cases little progress has been achieved. It was just recently that solutions were found for the more generic Bianchi class A models, in particular the Bianchi type IX model, resembling the situation that one faces in the full theory.

It was first remarked by Kodama (1988, 1990) that solutions to the Wheeler–DeWitt equation (WDW) in the formulation of Arnowitt, Deser, and Misner (ADM) and Ashtekar (in the connection representation) are related by $\Psi = \Psi_A e^{\pm i\Phi_A}$, where Φ_A is the homogeneous specialization of the generating functional (Ashtekar, 1986) of the canonical transformation from the ADM variables to Ashtekar's. This function was calculated explicitly for the

[1] Instituto de Física de la Universidad de Guanajuato, P.O. Box E-143, 37150, León, Gto, México.
[2] Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, P.O. Box 55-534, 09340, D. F., México.

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diagonal Bianchi type IX model by Kodama (1988, 1990), who also found Ψ_A = const as solution. Since Φ_A is pure imaginary, for a certain factor ordering, one expects a solution of the form $\Psi = We^{\pm\Phi}$, W = const, $\Phi = i\Phi_A$. In fact this type of solution has been found for the diagonal Bianchi type IX model (Moncrief and Ryan, 1991). For the special case of the Taub model (Martínez and Ryan, 1983; Moncrief and Ryan, 1991) it is also possible to find a solution for which $W = \text{const} \cdot e^{\alpha}e^{\beta_+}$.

In superquantum cosmology the same kind of solution has been found by means of two different approaches. Using supegravity $N = 1$ it was shown (D'Eath *et al.*, 1993) for the Bianchi type I model that the general solution has the form $\Psi = C_1 h^{-1/2} e^{-\Phi} + C_2 h^{-1/2} \psi^6 e^{+\Phi}$, where $C_1 = C_2$ = const, h is the determinant of the three-metric, and ψ^{2n} express symbolically the expansion of the wave function Ψ in even powers (this guarantees Lorentz invariance) of the gravitino field. The function Φ in this particular case is zero, but it was suggested that for the Bianchi class A models the solution has exactly this form with their corresponding Φ function. This conjecture has been confirmed in a series of publications using the ADM (D'Eath, 1993; Asano *et al.*, 1993) and the Ashtekar formulation (Capovilla and Guven, 1994; Capovilla and Obregón, 1994). A more general postulate for the Lorentz invariance allow solutions also in the ψ^2 and ψ^4 terms (Csordás and Graham, 1995). Similar solutions exist for a WDW equation derived for the bosonic sector of the heterotic string (Lidsey, 1994).

A second approach (Graham, 1991, 1993) considers the WDW equation also in the ADM and Ashtekar formulation (Obregón *et al.*, 1993a) for the Bianchi model of interest and proceeds by finding appropriate operators which are the "square root" of this equation. This procedure has the disadvantage that one has to introduce fermionic variables without a direct physical meaning. However, for the physical quantities of interest (like $\Psi^*\Psi$) one integrates over these variables (Obregón *et al.*, 1993b, 1994), getting information about their influence on the unnormalized probability function.

The three previous procedures virtually result in the same kind of quantum state and are of interest because for some of these models (e.g., Bianchi IX) these are the only known solutions. It is remarkable that they appear in the three different approaches mentioned. However, this kind of solution has been found in standard quantum cosmology only for the Bianchi type IX, Taub, and FRW models (Martínez and Ryan, 1983; Moncrief and Ryan, 1991). The main point of this paper is to generalize the results of Moncrief and Ryan to the diagonal Bianchi class A models. We will show that all solutions are of the form $\Psi = We^{\pm\Phi}$, where W is in general a function and can be reduced to a constant for the Bianchi models VIII and IX, depending on the factor ordering chosen in the WDW equation, and Φ is a solution to the corresponding classical Hamilton–Jacobi equation. For the Bianchi type

II and IX models there exist other real $\tilde{\Phi}_2$ and $\tilde{\Phi}_9$ solutions to the classical Hamilton–Jacobi equation; $\tilde{\Phi}_9$ is a solution in supersymmetric quantum cosmology (Csordás and Graham, 1995). However, we were unable to obtain this solution in standard quantum cosmology. It is possible that for the semi-general factor ordering (Hartle and Hawking, 1983) we have chosen, it is not possible to get this particular solution.

Let us recall here the canonical formulation in the ADM formalism of the diagonal Bianchi class A models. The metric has the form

$$ds^2 = -dt^2 + e^{2\alpha(t)}(e^{2\beta(t)})_{ij}\omega^i\omega^j \tag{1}$$

where $\alpha(t)$ is a scalar and $\beta_{ij}(t)$ a 3×3 diagonal matrix, $\beta_{ij} = \mathrm{diag}(x + \sqrt{3}y, x - \sqrt{3}y, -2x)$, and ω^i are one-forms that characterize each cosmological Bianchi type model, and which obey $d\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k$, with C^i_{jk} the structure constants of the corresponding invariance group.

The ADM action has the form

$$I = \int (P_x\,dx + P_y\,dy + P_\alpha\,d\alpha - N\mathcal{H}_\perp)\,dt \tag{2}$$

where

$$\mathcal{H}_\perp = e^{-3\alpha}(-P_\alpha^2 + P_x^2 + P_y^2 + e^{4\alpha}V(x, y)) \tag{3}$$

and $e^{4\alpha}V(x, y) = U(q^\mu)$ is the potential term of the cosmological model under consideration.

The WDW equation for these models is achieved by replacing P_{q^μ} by $-i\partial_{q^\mu}$ in (3), with $q^\mu = (\alpha, x, y)$. The factor $e^{-3\alpha}$ may be factor ordered with $\hat{P}_\alpha$ in many ways. Hartle and Hawking (1983) have suggested what might be called a semi-general factor ordering, which in this case would order $e^{-3\alpha}\hat{P}_\alpha^2$ as

$$-e^{-(3-Q)\alpha}\partial_\alpha e^{-Q\alpha}\partial_\alpha = -e^{-3\alpha}\partial_\alpha^2 + Qe^{-3\alpha}\partial_\alpha \tag{4}$$

where Q is any real constant. We will assume in the following this factor ordering for the Wheeler–DeWitt equation, which becomes

$$\Box\Psi + Q\frac{\partial\Psi}{\partial\alpha} - U(q^\mu)\Psi = 0 \tag{5}$$

where $\Box$ is the three-dimensional d'Alembertian in the q^μ coordinates, with signature $(-\ +\ +)$.

The paper is then organized as follows. In Section 2 we introduce the Ansatz $\Psi = We^{\pm\Phi}$ in (5) and set the general equations (under the assumed factor ordering) for the Bianchi class A models. In Section 3 we present solutions for the cosmological class A Bianchi models; only for the cases of

Bianchi VIII and IX can W be directly reduced to a constant. For all other Bianchi models W is in general a function. These solutions have the same form as those found in superquantum cosmology (Capovilla and Guven, 1994; Capovilla and Obregón, 1994). Section 4 is dedicated to final remarks.

2. TRANSFORMATION OF THE WHEELER–DEWITT EQUATION

Under the Ansatz for the wave function $\Psi(q^\mu) = W(\alpha, x, y)e^{-\Phi}$, (5) is transformed into

$$\Box W - W\Box\Phi - 2\nabla W\cdot\nabla\Phi + Q\frac{\partial W}{\partial\alpha} - QW\frac{\partial\Phi}{\partial\alpha} + W[(\nabla\Phi)^2 - U] = 0 \tag{6}$$

where

$$\Box = G^{\mu\nu}\frac{\partial^2}{\partial q^\mu \partial q^\nu}, \qquad \nabla W\cdot\nabla\Phi = G^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial\Phi}{\partial q^\nu}$$

$$(\nabla)^2 = -\left(\frac{\partial}{\partial\alpha}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$$

with $G^{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, and U is the potential term of the cosmological model under consideration.

If one can solve the nonlinear equation

$$(\nabla\Phi)^2 - U = 0 \tag{7}$$

for Φ, then one can obtain a master equation for the function W,

$$\Box W - W\Box\Phi - 2\nabla W\cdot\nabla\Phi + Q\frac{\partial W}{\partial\alpha} - QW\frac{\partial\Phi}{\partial\alpha} = 0 \tag{8}$$

Equation (7) is the classical Einstein–Hamilton–Jacobi equation, which can be obtained by replacing the momentum $P_{q^\mu} \to \partial\Phi/\partial q^\mu$ in (3).

We were able to solve (7) for the class A Bianchi models by making the change of coordinates $\beta_1 = \alpha + x + \sqrt{3}y$, $\beta_2 = \alpha + x - \sqrt{3}y$, and $\beta_3 = \alpha - 2x$ and rewriting (7) in these new coordinates. With this change, the function Φ is obtained and will be given in Section 3 in general form for the class A Bianchi models. In particular, Moncrief and Ryan (1991) found in the case of the Bianchi type IX model an exact solution for (7),

$$\Phi = \tfrac{1}{6}e^{2\alpha}[e^{-4x} + 2e^{2x}\cosh(2\sqrt{3}y)] \tag{9}$$

and then the solution for the wave function, where $W = \text{const}$, implying Q

$= -6$, and a solution for the Taub model, where the value of the Q parameter is zero and $W = \text{const} \cdot e^{\alpha+x}$.

In order to search for the particular solutions of interest, let us make the assumption

$$\Box W + Q \frac{\partial W}{\partial \alpha} = 0 \tag{10}$$

whose solution should be consistent with

$$W \Box \Phi + 2\nabla W \cdot \nabla \Phi + QW \frac{\partial \Phi}{\partial \alpha} = 0 \tag{11}$$

In the following we will study only the set of solutions for the separated system (10), (11), and not the whole set corresponding to (8) and also not to (5); then (10) is easier to solve than the original (5) because it does not contain any potential.

In the rest of this work, we restrict our analysis to different solutions for class A Bianchi models, where the Q parameter allows to consider a set of different factor orderings, restricted under the assumption (4), in the quantum Wheeler–DeWitt equation.

3. $\Psi = We^{\pm\Phi}$ SOLUTIONS

In this section we obtain the solutions to the equations that appear in the decomposition of the WDW equation, (7), (10), and (11), and give them for the class A cosmological Bianchi models.

Let us present, by means of a different procedure, the already known solution (Moncrief and Ryan, 1991) to (7) for the Bianchi type IX model, because using this procedure we were able to obtain solutions for the other Bianchi class A models.

Using the change of variables $(\alpha, x, y) \to (\beta_1, \beta_2, \beta_3)$, where the law of the transformation between the sets of variables is

$$\begin{aligned} \beta_1 &= \alpha + x + \sqrt{3}y \\ \beta_2 &= \alpha + x - \sqrt{3}y \\ \beta_3 &= \alpha - 2x \end{aligned} \tag{12}$$

we can write the equation $[\nabla]^2 = -(\partial/\partial\alpha)^2 + (\partial/\partial x)^2 + (\partial/\partial y)^2$ in the following way:

$$\begin{aligned} [\nabla]^2 &= 3\left[\left(\frac{\partial}{\partial\beta_1}\right)^2 + \left(\frac{\partial}{\partial\beta_2}\right)^2 + \left(\frac{\partial}{\partial\beta_3}\right)^2\right] - 6\left(\frac{\partial}{\partial\beta_1}\frac{\partial}{\partial\beta_2} + \frac{\partial}{\partial\beta_1}\frac{\partial}{\partial\beta_3} + \frac{\partial}{\partial\beta_2}\frac{\partial}{\partial\beta_3}\right) \\ &= 3\left(\frac{\partial}{\partial\beta_1} + \frac{\partial}{\partial\beta_2} + \frac{\partial}{\partial\beta_3}\right)^2 - 12\left(\frac{\partial}{\partial\beta_1}\frac{\partial}{\partial\beta_2} + \frac{\partial}{\partial\beta_1}\frac{\partial}{\partial\beta_3} + \frac{\partial}{\partial\beta_2}\frac{\partial}{\partial\beta_3}\right) \end{aligned} \tag{13}$$

The potential term of the Bianchi type IX model is transformed in the new variables as

$$U = \tfrac{1}{3}[(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3})^2 - 4e^{2(\beta_1+\beta_2)} - 4e^{2(\beta_1+\beta_3)} - 4e^{2(\beta_2+\beta_3)}] \tag{14}$$

Then (7) for this model is rewritten in the new variables as

$$3\left(\frac{\partial\Phi}{\partial\beta_1} + \frac{\partial\Phi}{\partial\beta_2} + \frac{\partial\Phi}{\partial\beta_3}\right)^2 - 12\left(\frac{\partial\Phi}{\partial\beta_1}\frac{\partial\Phi}{\partial\beta_2} + \frac{\partial\Phi}{\partial\beta_1}\frac{\partial\Phi}{\partial\beta_3} + \frac{\partial\Phi}{\partial\beta_2}\frac{\partial\Phi}{\partial\beta_3}\right) - \tfrac{1}{3}[(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3})^2 - 4e^{2(\beta_1+\beta_2)} - 4e^{2(\beta_1+\beta_3)} - 4e^{2(\beta_2+\beta_3)}] = 0 \tag{15}$$

Now we can use the separation-of-variables method to get solutions to the last equation for the Φ function, obtaining for the Bianchi type IX model (Moncrief and Ryan, 1991)

$$\Phi_9 = \pm\tfrac{1}{6}(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}) \tag{16}$$

Following this same procedure, it is possible to generalize this kind of solution to the rest of the Bianchi class A models. We show these results in Table I. With this result, the solution to (8) gives for the W function

$$W_9 = W_0 \exp[(3 + Q/2)\alpha] \tag{17}$$

where W_0 = const and $Q = \pm 6$; then the wave function has the form

$$\Psi_9 = W_0 \exp[(3 + Q/2)\alpha] \exp(\pm\Phi_9) \tag{18}$$

In the case of the Taub model, one replace in all terms only $y = P_y = 0$

$$\Phi_{\mathrm{TAUB}} = \tfrac{1}{6}e^{2\alpha}(2e^{2x} + e^{-4x}) \tag{19}$$

and the function W

$$W = W_0 \exp(\alpha + x) \tag{20}$$

In this last case, the only value of the Q parameter is zero. These solutions were given by Moncrief and Ryan (1991). In the case of FRW model, the

Table I. Potential U and Φ Function for Class A Bianchi Models

Bianchi type	Potential U	Φ
I	0	0
II	$\frac{1}{3}e^{4\beta_1}$	$\pm\frac{1}{6}e^{2\beta_1}$
$VI_{h=-1}$	$\frac{4}{3}e^{2(\beta_1+\beta_2)}$	$\pm\frac{1}{6}[2(\beta_1 - \beta_2)e^{(\beta_1+\beta_2)}]$
$VII_{h=0}$	$\frac{1}{3}[e^{4\beta_1} + e^{4\beta_2} - 2e^{2(\beta_1+\beta_2)}]$	$\pm\frac{1}{6}[e^{2\beta_1} + e^{2\beta_2}]$
VIII	$\frac{1}{3}[e^{4\beta_1} + e^{4\beta_2} + e^{4\beta_3} - 2e^{2(\beta_1+\beta_2)} + 2e^{2(\beta_1+\beta_3)} + 2e^{2(\beta_2+\beta_3)}]$	$\pm\frac{1}{6}[e^{2\beta_1} + e^{2\beta_2} - e^{2\beta_3}]$
IX	$\frac{1}{3}[e^{4\beta_1} + e^{4\beta_2} + e^{4\beta_3} - 2e^{2(\beta_1+\beta_2)} - 2e^{2(\beta_1+\beta_3)} - 2e^{2(\beta_2+\beta_3)}]$	$\pm\frac{1}{6}[e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}]$

value of $Q = 2$ and $W =$ const are obtained by means of this method. The solution is well known, $\Phi_{\text{FRW}} = \frac{1}{2}e^{2\alpha}$, and $\Psi_{\text{FRW}} = W_0 \exp(\pm\Phi_{\text{FRW}})$.

The functions W for the rest of the Bianchi class A models are shown in Table II.

If one looks at the expressions for the functions Φ_i, one notes that there exists a general form and we can write them using the 3×3 matrix m^{ij} that appears in the classification scheme of Ellis and MacCallum (1969; Ryan and Shepley, 1975); the structure constants are written in the form

$$C^i_{jk} = \epsilon_{jks}m^{si} + \delta^i_{[k}a_{j]} \tag{21}$$

where $a_i = 0$ for the class A models.

If we define $g_i(q^\mu) = (e^{\beta_1}, e^{\beta_2}, e^{\beta_3})$, with β_i given in (12), all solutions to (7) can be written as

$$\Phi(q^\mu) = \pm\tfrac{1}{6}[g_i M^{ij}(g_j)^T] \tag{22}$$

where $M^{ij} = m^{ij}$ for the class A Bianchi models, except for the Bianchi type $\text{VI}_{h=-1}$, for which we redefine the matrix to be consistent with (22):

$$M^{ij} = \frac{6}{\sqrt{3}}y\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

For the rest of the models (22) can be reduced to the expression given in the literature in connection with superquantum cosmology (Asano *et al.*, 1993; Lidsey, 1994)

$$\Phi(q^\mu) = \pm\tfrac{1}{6}(m^{ij}g_{ij}) \tag{23}$$

where g_{ij} is the 3-metric. Then, for the class A Bianchi models the wave function Ψ can be written in the general form

$$\Psi = W\exp\{\pm\tfrac{1}{6}[g_i M^{ij}(g_j)^T]\} \tag{24}$$

and for each cosmological model under consideration the wave function of interest can be read from Tables I and II.

Table II. *W* Function and Constraints Between the Constants in the Solutions for Class A Bianchi Models

Bianchi type	W	Constraint
I	$\exp(\mathbf{x}\cdot\mathbf{k})$	$a^2 - aQ - (b^2 + c^2) = 0$
II	$\exp[(3 + Q/2 - a)\alpha + (b - a)x - (b/\sqrt{3})y]$	$108 - 72a + 24ab - 16b^2 - 3Q^2 = 0$
$\text{VI}_{h=-1}$	$\exp[\frac{1}{4}(\alpha + x)]$	$Q = 0$
$\text{VII}_{h=0}$	$\exp[(3 + Q/2 - a)\alpha - ax]$	$36 - 24a - Q^2 = 0$
VIII	$\exp[(3 + Q/2)\alpha]$	$Q = \pm 6$
IX	$\exp[(3 + Q/2)\alpha]$	$Q = \pm 6$

4. FINAL REMARKS

Wave functions of the form $\Psi = We^{\pm\Phi}$ are the only known exact solutions for the Bianchi type IX model in standard quantum cosmology. This kind of solution already has been found in supersymmetric quantum cosmology and also for the WDW equation defined in the bosonic sector of the heterotic strings. We have shown that they are also exact solutions to the rest of the class A Bianchi models in standard quantum cosmology, under the assumed semi-general factor ordering (4). Different procedures seem to produce this particular quantum state, where Φ is a solution to the corresponding classical Hamilton–Jacobi equation (7).

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